

Canonical Quantization of SU(3) Skyrme Model in a General Representation

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Abstract

A complete canonical quantization of the SU(3) Skyrme model performed in the collective coordinate formalism in general irreducible representations. In the case of SU(3) the model differs qualitatively in different representations. The Wess-Zumino-Witten term vanishes in all self-adjoint representations in the collective coordinate method for separation of space and time variables. The canonical quantization generates representation dependent quantum mass corrections, which can stabilize the soliton solution. The standard symmetry breaking mass term, which in general leads to representation mixing, degenerates to the SU(2) form in all self-adjoint representations.

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1. INTRODUCTION

The Skyrme model is a nonlinear field theory, with localized finite energy soliton solutions, which may be quantized as fermions [1, 2]. The semi-classically quantized Skyrme model has proven useful for baryon phenomenology as a realization of the large color limit of QCD [3]. The original model was defined for a unitary field $U(\mathbf{x}, t)$ that belongs to fundamental representation of $SU(2)$. The boundary condition $U \rightarrow \mathbb{1}$ as $|\mathbf{x}| \rightarrow \infty$ implies that the unitary field represents a mapping from $S^3 \rightarrow S^3$, the integer valued winding number of which classifies the solitonic sectors of the model and may interpreted as the baryon number. The model has subsequently been directly generalized the $SU(3)$ and $SU(N)$ [4], in which case the field $U(\mathbf{x}, t)$ is described by group valued functions with semiclassical quantization.

Both the $SU(2)$ and $SU(3)$ Skyrme versions of the model have been quantized canonically in refs. [5, 6] in the collective coordinate formalism. The canonical quantization procedure leads to quantum corrections to the skyrmion mass, which restore the stability of the soliton solutions that is lost in the semiclassical quantization. This method has subsequently been generalized to unitary fields $U(\mathbf{x}, t)$ that belong to general representations of the $SU(2)$ [7, 8, 9], along with a demonstration that the quantum corrections, which stabilize the soliton solutions, are representation dependent.

The aim of the present paper is to extend the canonically quantized Skyrme model to general irreducible representations (irrep) of $SU(3)$. The motivation is the absence of any *a priori* reason to restrict collective chiral models to the fundamental representation of the group. The focus here is on the mathematical aspects of the model, and on the derivation of both the Hamiltonian density and the Hamiltonian, in order to elucidate their representation dependence. The possible phenomenological applications both in hyperon and hypernuclear phenomenology as well as in the Skyrme model description of the quantum Hall effect [10] and Bose-Einstein condensates [11], are not elaborated.

In contrast to the case of $SU(2)$, the solutions to the $SU(3)$ Skyrme model depend in an essential way on the dimension. Remarkably the Wess-Zumino-Witten (WZW) term vanishes in all self-adjoint irreps of $SU(3)$, as it is proportional to the cubic Casimir operator $C_3^{SU(3)}$ in the collective coordinate method for separation of the dependence on space and time variables. In the self adjoint irreps the symmetry breaking mass term in the model reduces to the $SU(2)$ form.

After some preliminary definitions in Section 2 below, the main part of this paper is organized as follows. In Section 3, the classical treatment of the Skyrme model in a general irrep of $SU(3)$ is reviewed. In Section 4, the quantum Skyrme model is constructed *ab initio* in the collective coordinates framework. In Section 5, the WZW term is taken into account and the left and right transformation generators are derived from the Lagrangian. The Lagrangian and Hamiltonian density operators are given explicitly in terms of generators. In Section 6, the symmetry breaking term are considered in the collective coordinates framework. Section 7 contains a summarizing discussion. A number of relevant mathematical details are given in the Appendix.

2. DEFINITIONS FOR THE UNITARY $SU(3)$ SOLITON FIELD

The unitary field $U(\mathbf{x}, t)$ is defined for for general irreps (λ, μ) of $SU(3)$ in addition to the fundamental representation $(1, 0)$. The related Young tableaux are denoted $[\lambda_1, \lambda_2, \lambda_3]$, where $\lambda = \lambda_1 - \lambda_2$, $\mu = \lambda_2 - \lambda_3$. A group element is specified by the eight real parameters $\alpha^i(\mathbf{x}, t)$. The unitary field is expressed in the form of Wigner D matrices for $SU(3)$ in (λ, μ) irrep as:

$$U(\mathbf{x}, t) = D^{(\lambda, \mu)}(\alpha^i(\mathbf{x}, t)). \quad (2.1)$$

The one-form of the unitary field belongs to the Lie algebra of $SU(3)$. The one-forms may be determined by the functions $C_i^{(Z, I, M)}(\alpha)$ and $C'_i{}^{(Z, I, M)}(\alpha)$, the explicit expressions for which depend on the specific group parameterization:

$$\begin{aligned} \partial_i U U^\dagger &= \left(\frac{\partial}{\partial \alpha^i} U \right) U^\dagger = C_i^{(Z, I, M)}(\alpha) \left\langle \left| J_{(Z, I, M)}^{(1, 1)} \right| \right\rangle, \\ U^\dagger \partial_i U &= U^\dagger \frac{\partial}{\partial \alpha^i} U = C'_i{}^{(Z, I, M)}(\alpha) \left\langle \left| J_{(Z, I, M)}^{(1, 1)} \right| \right\rangle. \end{aligned} \quad (2.2)$$

The parameters spin I , and its projections M and Z , which is related to hypercharge as $Y = -2Z$, specify the basis states of irrep $(1, 1)$.

The parameterization for the $SU(3)$ model, and the expressions of the differential Casimir operator in terms of the Euler angles, has been proposed by Yabu and Ando [12]. The $SU(3)$ generators are defined as components of irreducible tensors $(1, 1)$ and may be

expanded in terms of the Gell-Mann generators Λ_k :

$$\begin{aligned}
J_{(0,0,0)}^{(1,1)} &= -\frac{1}{2}\Lambda_8, & J_{(0,1,0)}^{(1,1)} &= \frac{1}{2}\Lambda_3, \\
J_{(0,1,1)}^{(1,1)} &= -\frac{1}{2\sqrt{2}}(\Lambda_1 + i\Lambda_2), & J_{(0,1,-1)}^{(1,1)} &= \frac{1}{2\sqrt{2}}(\Lambda_1 - i\Lambda_2), \\
J_{(-\frac{1}{2},\frac{1}{2},\frac{1}{2})}^{(1,1)} &= \frac{1}{2\sqrt{2}}(\Lambda_4 + i\Lambda_5), & J_{(-\frac{1}{2},\frac{1}{2},-\frac{1}{2})}^{(1,1)} &= \frac{1}{2\sqrt{2}}(\Lambda_6 + i\Lambda_7), \\
J_{(\frac{1}{2},\frac{1}{2},\frac{1}{2})}^{(1,1)} &= -\frac{1}{2\sqrt{2}}(\Lambda_6 - i\Lambda_7), & J_{(\frac{1}{2},\frac{1}{2},-\frac{1}{2})}^{(1,1)} &= \frac{1}{2\sqrt{2}}(\Lambda_4 - i\Lambda_5).
\end{aligned} \tag{2.3}$$

In the case of the fundamental representation the Λ_k matrices generators reduce to the standard Gell-Mann matrices λ_i . Although the generators (2.3) are non-hermitian: $(J_{(Z,I,M)}^{(1,1)})^\dagger = (-1)^{Z+M} J_{(-Z,I,-M)}^{(1,1)}$, the commutation relations nevertheless have the simple form:

$$\left[J_{(Z',I',M')}^{(1,1)}, J_{(Z'',I'',M'')}^{(1,1)} \right] = \sum_{(Z,I,M)} -\sqrt{3} \begin{bmatrix} (1,1) & (1,1) & (1,1)_a \\ (Z',I',M') & (Z'',I'',M'') & (Z,I,M) \end{bmatrix} J_{(Z,I,M)}^{(1,1)}. \tag{2.4}$$

Here the coefficient on the r.h.s. of eq. (2.4) is a Clebsch-Gordan coefficient of SU(3), the explicit expressions for which are given in [13]. The index a in the Clebsch-Gordan coefficient denotes that only antisymmetric irrep couplings are included.

For the specification of the basis states in a general irrep (λ, μ) the parameters (z, j, m) , where the hypercharge is $y = \frac{2}{3}(\mu - \lambda) - 2z$, are employed. The basis state parameters satisfy the inequalities:

$$\begin{aligned}
j - m &\geq 0, & j - z &\geq 0, \\
j + m &\geq 0, & j + z &\geq 0, \\
\lambda + z - j &\geq 0, & \mu - z - j &\geq 0,
\end{aligned} \tag{2.5}$$

where the left hand sides are integers. The generators (2.3) act on the basis states as

follows:

$$\begin{aligned}
J_{(0,0,0)}^{(1,1)} \left| \begin{matrix} (\lambda, \mu) \\ z, j, m \end{matrix} \right\rangle &= -\frac{\sqrt{3}}{2} y \left| \begin{matrix} (\lambda, \mu) \\ z, j, m \end{matrix} \right\rangle, & J_{(0,1,-1)}^{(1,1)} \left| \begin{matrix} (\lambda, \mu) \\ z, j, m \end{matrix} \right\rangle &= \sqrt{\frac{1}{2}(j+m)(j-m+1)} \left| \begin{matrix} (\lambda, \mu) \\ z, j, m-1 \end{matrix} \right\rangle, \\
J_{(0,1,0)}^{(1,1)} \left| \begin{matrix} (\lambda, \mu) \\ z, j, m \end{matrix} \right\rangle &= m \left| \begin{matrix} (\lambda, \mu) \\ z, j, m \end{matrix} \right\rangle, & J_{(0,1,1)}^{(1,1)} \left| \begin{matrix} (\lambda, \mu) \\ z, j, m \end{matrix} \right\rangle &= -\sqrt{\frac{1}{2}(j-m)(j+m+1)} \left| \begin{matrix} (\lambda, \mu) \\ z, j, m+1 \end{matrix} \right\rangle, \\
J_{(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2})}^{(1,1)} \left| \begin{matrix} (\lambda, \mu) \\ z, j, m \end{matrix} \right\rangle &= \sqrt{\frac{(\lambda+z-j)(\mu-z+j+2)(j-z+1)(j+m+1)}{2(2j+1)(2j+2)}} \left| \begin{matrix} (\lambda, \mu) \\ z - \frac{1}{2}, j + \frac{1}{2}, m + \frac{1}{2} \end{matrix} \right\rangle \\
&\quad - \sqrt{\frac{(\lambda+z+j+1)(\mu-z-j+1)(j+z)(j-m)}{4j(2j+1)}} \left| \begin{matrix} (\lambda, \mu) \\ z - \frac{1}{2}, j - \frac{1}{2}, m + \frac{1}{2} \end{matrix} \right\rangle, \\
J_{(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})}^{(1,1)} \left| \begin{matrix} (\lambda, \mu) \\ z, j, m \end{matrix} \right\rangle &= -\sqrt{\frac{(\lambda+z+j+2)(\mu-z-j)(z+j+1)(j-m+1)}{2(2j+1)(2j+2)}} \left| \begin{matrix} (\lambda, \mu) \\ z + \frac{1}{2}, j + \frac{1}{2}, m - \frac{1}{2} \end{matrix} \right\rangle \\
&\quad + \sqrt{\frac{(\lambda+z-j+1)(\mu-z+j+1)(j-z)(j+m)}{4j(2j+1)}} \left| \begin{matrix} (\lambda, \mu) \\ z + \frac{1}{2}, j - \frac{1}{2}, m - \frac{1}{2} \end{matrix} \right\rangle, \\
J_{(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})}^{(1,1)} \left| \begin{matrix} (\lambda, \mu) \\ z, j, m \end{matrix} \right\rangle &= \sqrt{\frac{(\lambda+z-j)(\mu-z+j+2)(j-z+1)(j-m+1)}{2(2j+1)(2j+2)}} \left| \begin{matrix} (\lambda, \mu) \\ z - \frac{1}{2}, j + \frac{1}{2}, m - \frac{1}{2} \end{matrix} \right\rangle \\
&\quad + \sqrt{\frac{(\lambda+z+j+1)(\mu-z-j+1)(j+z)(j+m)}{4j(2j+1)}} \left| \begin{matrix} (\lambda, \mu) \\ z - \frac{1}{2}, j - \frac{1}{2}, m - \frac{1}{2} \end{matrix} \right\rangle, \\
J_{(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})}^{(1,1)} \left| \begin{matrix} (\lambda, \mu) \\ z, j, m \end{matrix} \right\rangle &= -\sqrt{\frac{(\lambda+z+j+2)(\mu-z-j)(j+z+1)(j+m+1)}{2(2j+1)(2j+2)}} \left| \begin{matrix} (\lambda, \mu) \\ z + \frac{1}{2}, j + \frac{1}{2}, m + \frac{1}{2} \end{matrix} \right\rangle \\
&\quad - \sqrt{\frac{(\lambda+z-j+1)(\mu-z+j+1)(j-z)(j-m)}{4j(2j+1)}} \left| \begin{matrix} (\lambda, \mu) \\ z + \frac{1}{2}, j - \frac{1}{2}, m + \frac{1}{2} \end{matrix} \right\rangle.
\end{aligned} \tag{2.6}$$

The basis states are chosen such that the generators $J_{(0,1,0)}^{(1,1)}$ and $J_{(0,0,0)}^{(1,1)}$, as well as the Casimir operator of the $SU(2)$ subgroup $\hat{C}^{SU(2)} = \sum (-1)^M J_{(0,1,M)}^{(1,1)} J_{(0,1,-M)}^{(1,1)}$, are diagonal and thus provide a labelling of the basis states:

$$\hat{C}^{SU(2)} \left| \begin{matrix} (\lambda, \mu) \\ z, j, m \end{matrix} \right\rangle = j(j+1) \left| \begin{matrix} (\lambda, \mu) \\ z, j, m \end{matrix} \right\rangle. \tag{2.7}$$

3. THE CLASSICAL $SU(3)$ SKYRME MODEL

The action of the Skyrme model in $SU(3)$ is taken to have the form:

$$S = \int d^4x (\mathcal{L}_{Sk} + \mathcal{L}_{SB}) + S_{WZ}, \tag{3.8}$$

where the chirally symmetric Lagrangian density is [1]

$$\mathcal{L}_{Sk} = -\frac{f_\pi^2}{4} \text{Tr}\{\mathbf{R}_\mu \mathbf{R}^\mu\} + \frac{1}{32e^2} \text{Tr}\{[\mathbf{R}_\mu, \mathbf{R}_\nu][\mathbf{R}^\mu, \mathbf{R}^\nu]\}. \tag{3.9}$$

Here the right chiral current is defined as

$$\mathbf{R}_\mu = (\partial_\mu U) U^\dagger = \partial_\mu \alpha^i C_i^{(A)}(\alpha) \left\langle \left| J_{(A)}^{(1,1)} \right| \right\rangle. \quad (3.10)$$

The Greek characters indicate differentiation with respect to spacetime variables $\partial_\mu = \partial/\partial x^\mu$ in the metric $\text{diag}(\eta_{\alpha\beta}) = (1, -1, -1, -1)$. The only parameters of the model are f_π and e . The symmetry breaking term \mathcal{L}_{SB} and Wess-Zumino-Witten action S_{WZ} are specified below.

Upon substitution of (3.10) into (3.9) the classical Lagrangian density may be expressed in terms of the group parameters α^i as:

$$\begin{aligned} \mathcal{L}_{Sk} = & \frac{3}{32N} \dim(\lambda, \mu) C_2^{SU(3)}(\lambda, \mu) \left\{ -f_\pi^2 (-1)^A \partial_\mu \alpha^i C_i^{(A)}(\alpha) \partial^\mu \alpha^{i'} C_{i'}^{(-A)}(\alpha) \right. \\ & + \frac{3}{8e^2} \cdot (-1)^A \partial_\mu \alpha^i C_i^{(A^1)}(\alpha) \partial_\nu \alpha^{i'} C_{i'}^{(A^2)}(\alpha) \\ & \times \partial^\mu \alpha^{i''} C_{i''}^{(A^3)}(\alpha) \partial^\nu \alpha^{i'''} C_{i'''}^{(A^4)}(\alpha) \left[\begin{array}{ccc} (1,1) & (1,1) & (1,1)_a \\ (A^1) & (A^2) & (A) \end{array} \right] \left[\begin{array}{ccc} (1,1) & (1,1) & (1,1)_a \\ (A^3) & (A^4) & (-A) \end{array} \right] \left. \right\}. \end{aligned} \quad (3.11)$$

In the last $SU(3)$ Clebsch-Gordan coefficients only the antisymmetric irrep coupling is included and there is no summation over irrep multiplicity. The capital Latin character indices (A) denote the state label (Z, I, M) , $(-A)$ denotes $(-Z, I, -M)$ and $(-1)^A = (-1)^{Z+M}$. The dependence on group irrep appear as an overall factor because

$$\text{Tr} \left\langle (\lambda, \mu) \left| J_{(A)}^{(1,1)} J_{(B)}^{(1,1)} \right| (\lambda, \mu) \right\rangle = (-1)^A \frac{1}{8} \dim(\lambda, \mu) C_2^{SU(3)}(\lambda, \mu) \delta_{(A), (-B)}, \quad (3.12)$$

where $\dim(\lambda, \mu) = \frac{1}{2}(\lambda+1)(\mu+1)(\lambda+\mu+2)$ is a dimension of irrep. Above $C_2^{SU(3)}(\lambda, \mu) = \frac{1}{3}(\lambda^2 + \mu^2 + \lambda\mu + 3\lambda + 3\mu)$ is an eigenvalue of the quadratic Casimir operator of $SU(3)$:

$$\hat{C}_2^{SU(3)}(\lambda, \mu) = (-1)^{Z+M} J_{(Z, I, M)}^{(1,1)} J_{(-Z, I, -M)}^{(1,1)}. \quad (3.13)$$

The time component of the conserved topological current in the Skyrme model represents the baryon number density which in terms of the variables $\alpha^i(x, t)$ takes the form:

$$\begin{aligned} B^0(x) = & \frac{1}{24\pi^2 N} \epsilon^{0ikl} \text{Tr} (\partial_i U) U^\dagger (\partial_k U) U^\dagger (\partial_l U) U^\dagger \\ = & \frac{(-1)^A}{2^7 \sqrt{3} \pi^2 N} \dim(\lambda, \mu) C_2^{SU(3)}(\lambda, \mu) \epsilon^{abc} \partial_a \alpha^i C_i^{(A)}(\alpha) \\ & \times \partial_b \alpha^{i'} C_{i'}^{(A')}(\alpha) \partial_c \alpha^{i''} C_{i''}^{(A'')}(\alpha) \left[\begin{array}{ccc} (1,1) & (1,1) & (1,1)_a \\ (A') & (A'') & (-A) \end{array} \right]. \end{aligned} \quad (3.14)$$

For the classical chiral symmetric Skyrme model the dependence on the irrep is contained in the overall factor N . The normalization factor,

$$N = \frac{1}{4} \dim(\lambda, \mu) C_2^{SU(3)}(\lambda, \mu), \quad (3.15)$$

is chosen so as to that the smallest non trivial baryon number equals unity: $B = \int d^3x B^0(x) = 1$. The dynamics of the system are independent of the overall factor in Lagrangian. Therefore in the classical case the Skyrme model defined in arbitrary irrep is equivalent to the Skyrme model in the fundamental representation $(1, 0)$, for which $N = 1$.

The classical soliton solution of the hedgehog type for (λ, μ) irrep of the $SU(3)$ group may be expressed as direct sum of hedgehog ansätze in $SU(2)$ irreps [8]. The $SU(2)$ representations embedded in the (λ, μ) irrep are defined by the canonical $SU(3) \supset SU(2)$ chain. The hedgehog generalization takes the form:

$$\exp i(\sigma \cdot \hat{x})F(r) \rightarrow U_0(\hat{x}, F(r)) = \exp i2 \left(J_{(0,1,\cdot)}^{(1,1)} \cdot \hat{x} \right) F(r) = \sum_{z,j}^{(\lambda,\mu)} \oplus D^j(\mathcal{K}), \quad (3.16)$$

where σ are Pauli matrices and \hat{x} is the unit vector. The Euler angles of the $SU(2)$ subgroup in terms of polar angles φ, θ and the chiral angle function $F(r)$ are:

$$\begin{aligned} \mathcal{K}^1 &= \varphi - \arctan(\cos \theta \tan F(r)) - \pi/2, \\ \mathcal{K}^2 &= -2 \arcsin(\sin \theta \sin F(r)), \\ \mathcal{K}^3 &= -\varphi - \arctan(\cos \theta \tan F(r)) + \pi/2. \end{aligned} \quad (3.17)$$

The normalization factor (3.15) ensures that the baryon number density for the hedgehog skyrmion in a general irrep has the usual form:

$$\begin{aligned} B^0(x) &= \frac{1}{24\pi^2 N} \epsilon^{0ikl} \text{Tr} (\partial_i U_0) U_0^\dagger (\partial_k U_0) U_0^\dagger (\partial_l U_0) U_0^\dagger \\ &= -\frac{1}{2\pi^2} \frac{\sin^2 F(r)}{r^2} F'(r). \end{aligned} \quad (3.18)$$

With the hedgehog ansatz (3.16), and after renormalization with the factor (3.15), the Lagrangian density (3.9) reduces to the following simple form:

$$\begin{aligned} \mathcal{L}_{cl}[F(r)] &= -\mathcal{M}_{cl}(F(r)) = -\left\{ \frac{f_\pi^2}{2} \left(F'^2 + \frac{2}{r^2} \sin^2 F \right) \right. \\ &\quad \left. + \frac{1}{8e^2} \frac{\sin^2 F}{r^2} \left(2F'^2 + \frac{\sin^2 F}{r^2} \right) \right\}. \end{aligned} \quad (3.19)$$

Variation of the classical hedgehog soliton mass leads to standard differential equation for the chiral angle $F(r)$.

The $SU(3)$ chiral symmetry breaking term of Lagrangian density is defined here as:

$$\mathcal{L}_{SB} = -\mathcal{M}_{SB} = -\frac{1}{N} \frac{f_\pi^2}{4} \left[m_0^2 \text{Tr} \{ U + U^\dagger - 2\mathbf{1} \} - 2m_8^2 \text{Tr} \left\{ (U + U^\dagger) J_{(0,0,0)}^{(1,1)} \right\} \right]. \quad (3.20)$$

This form is chosen so that it reduces to the mass term of the π, K, η mesons when the unitary field $U(\mathbf{x}, t) = \exp[\frac{i}{f_\pi} \varphi_k \Lambda_k]$ is expanded around the classical vacuum $U = \mathbf{1}$:

$$\mathcal{L}_{SB} = -\frac{1}{2} m_\pi^2 (\varphi_1^2 + \varphi_2^2 + \varphi_3^2) - \frac{1}{2} m_K^2 (\varphi_4^2 + \varphi_5^2 + \varphi_6^2 + \varphi_7^2) - \frac{1}{2} m_\eta^2 \varphi_8^2 + \dots \quad (3.21)$$

For arbitrary irrep the coefficients in the symmetry breaking term can be readily obtained as:

$$m_0^2 = \frac{1}{3} (m_\pi^2 + 2m_K^2), \quad m_8^2 = \frac{10}{3\sqrt{3}} \frac{C_2^{SU(3)}(\lambda, \mu)}{C_3^{SU(3)}(\lambda, \mu)} (m_\pi^2 - m_K^2), \quad (3.22)$$

where

$$C_3^{SU(3)}(\lambda, \mu) = \frac{1}{9} (\lambda - \mu)(2\lambda + \mu + 3)(2\mu + \lambda + 3), \quad (3.23)$$

is the eigenvalue of the cubic Casimir operator of $SU(3)$.

For the self adjoint irreps $\lambda = \mu$ the symmetry breaking part of Lagrangian (3.20) is proportional to $m_0^2 = \frac{1}{4} m_\pi^2$ only. The Gell-Mann-Okubo mass formula:

$$m_\pi^2 + 3m_\eta^2 - 4m_K^2 = 0, \quad (3.24)$$

is satisfied in all but the self adjoint irreps.

4. QUANTIZATION OF THE SKYRMION

The direct quantization of the Skyrme model even in the case of $SU(2)$ leads to rather complicated equations [7]. Here the collective coordinates [3] for the unitary field U in (λ, μ) irrep are employed for the separation of the variables, which depend on the temporal and spatial coordinates:

$$U(\hat{x}, F(r), \mathbf{q}(t)) = A(\mathbf{q}(t)) U_0(\hat{x}, F(r)) A^\dagger(\mathbf{q}(t)). \quad (4.25)$$

Because of form of the ansatz U_0 (3.16), the unitary field U is invariant under right $U(1)$ transformation of the $A(\mathbf{q}(t)) = D^{(\lambda,\mu)}(\mathbf{q}(t))$ matrix, defined as

$$A(\mathbf{q}(t)) \rightarrow A(\mathbf{q}(t)) \exp \beta J_{(0,0,0)}^{(1,1)}. \quad (4.26)$$

Thus the seven-dimensional homogeneous space $SU(3)/U(1)$, which is specified by the seven real, independent parameters $q^k(t)$, has to be considered. The mathematical structure of the Skyrme model and its quantization problems on the coset space $SU(3)/U(1)$ have been examined by several authors [14, 16]. The canonical quantization procedure for the $SU(3)$ Skyrme model in the fundamental representation has been considered by Fujii *et al.* [6]. Here the attention is on the representation dependence of the model. The Lagrangian (3.9) is considered quantum mechanically *ab initio*. The generalized coordinates $q^k(t)$ and velocities $(d/dt)q^k(t) = \dot{q}^k(t)$ satisfy the commutation relations:

$$[\dot{q}^k, q^l] = -i f^{kl}(q), \quad (4.27)$$

where $f^{kl}(q)$ are functions only of q^k , and the form of which will be determined below. The commutation relation between a velocity component \dot{q}^k and arbitrary function $G(q)$ is given by

$$[\dot{q}^k, G(q)] = -i \sum_r f^{kr}(q) \partial_r G(q). \quad (4.28)$$

For the time derivative the usual Weyl ordering is adopted:

$$\partial_0 G(q) = \frac{1}{2} \left\{ \dot{q}^k, \frac{\partial}{\partial q^k} G(q) \right\}. \quad (4.29)$$

The operator ordering is fixed by the form of the Lagrangian (3.9), without further ordering ambiguity.

The ansatz (4.25) is then substituted in the Skyrme Lagrangian (3.9) followed by an integration over the spatial coordinates. The Lagrangian is then obtained in terms of collective coordinates and velocities. For the derivation of the canonical momenta it is sufficient to restrict the consideration to terms of second order in the velocities here (the

terms of first order vanish). This leads to:

$$\begin{aligned}
L_{Sk} &\approx - \int dr r^2 \left[\sum_M (-1)^M \left\{ \dot{q}^i, C_i'^{(0,1,M)}(q) \right\} \left\{ \dot{q}^{i'}, C_{i'}'^{(0,1,-M)}(q) \right\} \right. \\
&\quad \times \frac{\pi}{3} \sin^2 F \left(f_\pi^2 + \frac{1}{e^2} \left(F'^2 + \frac{1}{r^2} \sin^2 F \right) \right) \\
&\quad + \sum_{Z,M} (-1)^{Z+M} \left\{ \dot{q}^i, C_i'^{(Z,\frac{1}{2},M)}(q) \right\} \left\{ \dot{q}^{i'}, C_{i'}'^{(-Z,\frac{1}{2},-M)}(q) \right\} \\
&\quad \times \frac{\pi}{4} (1 - \cos F) \left(f_\pi^2 + \frac{1}{4e^2} \left(F'^2 + \frac{2}{r^2} \sin^2 F \right) \right) \Big] \\
&\approx \frac{1}{2} \dot{q}^\alpha g_{\alpha\beta}(q, F) \dot{q}^\beta + [(\dot{q})^0 \text{--order term}] \\
&\approx \frac{1}{8} \left\{ \dot{q}^\alpha, C_\alpha'^{(A)}(q) \right\} E_{(A)(B)}(F) \left\{ \dot{q}^\beta, C_\beta'^{(B)}(q) \right\} + [(\dot{q})^0 \text{--order term}]. \quad (4.30)
\end{aligned}$$

The Lagrangian (4.30) is normalized by the factor (3.15). The metric tensor takes the form

$$g_{\alpha\beta}(q, F) = C_\alpha'^{(A)}(q) E_{(A)(B)}(F) C_\beta'^{(B)}(q). \quad (4.31)$$

where

$$E_{(Z,I,M)(Z',I',M')}(F) = -(-1)^{Z+M} a_I(F) \delta_{Z,-Z'} \delta_{I,I'} \delta_{M,-M'}. \quad (4.32)$$

Here the soliton moments of inertia are given as integrals over the dimensionless variable $\tilde{r} = e f_\pi r$:

$$a_0(F) = 0, \quad (4.33a)$$

$$a_{\frac{1}{2}}(F) = \frac{1}{e^3 f_\pi} \tilde{a}_{\frac{1}{2}}(F) = \frac{1}{e^3 f_\pi} 2\pi \int d\tilde{r} \tilde{r}^2 (1 - \cos F) \left[1 + \frac{1}{4} F'^2 + \frac{1}{2\tilde{r}^2} \sin^2 F \right], \quad (4.33b)$$

$$a_1(F) = \frac{1}{e^3 f_\pi} \tilde{a}_1(F) = \frac{1}{e^3 f_\pi} \frac{8\pi}{3} \int d\tilde{r} \tilde{r}^2 \sin^2 F \left[1 + F'^2 + \frac{1}{r^2} \sin^2 F \right]. \quad (4.33c)$$

The canonical momentum, which is conjugate to q^β , is defined as

$$p_\beta^{(0)} = \frac{\partial L_{Sk}}{\partial \dot{q}^\beta} = \frac{1}{2} \{ \dot{q}^\alpha, g_{\alpha\beta} \}. \quad (4.34)$$

The canonical commutation relations

$$\begin{aligned}
[q^\alpha, q^\beta] &= [p_\alpha^{(0)}, p_\beta^{(0)}] = 0, \\
[p_\beta^{(0)}, q^\alpha] &= -i\delta_{\alpha\beta}, \quad (4.35)
\end{aligned}$$

then yield the following explicit form for the functions $f^{\alpha\beta}(q)$:

$$f^{\alpha\beta}(q) = (g_{\alpha\beta})^{-1} = C_{(\bar{A})}^\alpha(q) E^{(\bar{A})(\bar{B})}(F) C_{(\bar{B})}^\beta(q), \quad (4.36)$$

where

$$E^{(\overline{Z}, \overline{I}, \overline{M})(\overline{Z}', \overline{I}', \overline{M}')} (F) = -(-1)^{Z+M} \frac{1}{a_I(F)} \delta_{Z, -Z'} \delta_{I, I'} \delta_{M, -M'} . \quad (4.37)$$

Note that here $E^{(0)(0)}(F)$ is left undefined. The summation over the indices (\bar{A}) denotes summation over the basis states (Z, I, M) of irrep $(1, 1)$, excluding the state $(0, 0, 0)$. It proves convenient to introduce the reciprocal function matrix $C'_{(\bar{A})}{}^\alpha(q)$, the properties of which are described in the Appendix. The commutation relations of the momenta (4.35) ensure the choice of parameters q^α on the manifold $SU(3)/U(1)$ (see [5]). Here there is no need for explicit parameterization of q^α .

After determination of function $f^{\alpha\beta}(q)$ the following explicit expression $A^\dagger \dot{A}$ obtains:

$$\begin{aligned} A^\dagger \dot{A} &= \frac{1}{2} D^{(\lambda, \mu)}(-q) \{ \dot{q}^\alpha, \partial_\alpha D^{(\lambda, \mu)}(q) \} \\ &= \frac{1}{2} \{ \dot{q}^\alpha, C'_\alpha{}^{(A)}(q) \} \left\langle \left| J_{(\bar{A})}^{(1,1)} \right| \right\rangle \\ &\quad - \frac{1}{2} i E^{(\bar{A})(\bar{B})}(F) C'_{(\bar{B})}{}^\beta(q) C'_\beta{}^{(0)}(q) \left(\left\langle \left| J_{(0)}^{(1,1)} J_{(\bar{A})}^{(1,1)} \right| \right\rangle + \left\langle \left| J_{(\bar{A})}^{(1,1)} J_{(0)}^{(1,1)} \right| \right\rangle \right) \\ &\quad - \frac{3}{8} i E^{(\bar{A})(\bar{B})}(F) C'_{(\bar{A})}{}^\alpha(q) C'_\alpha{}^{(0)}(q) C'_{(\bar{B})}{}^\beta(q) C'_\beta{}^{(0)}(q) \sum_{z,j}^{(\lambda, \mu)} \oplus y^2 \mathbb{1}_{z,j} \\ &\quad + \frac{i}{2a_{\frac{1}{2}}(F)} C_2^{SU(3)}(\lambda, \mu) \mathbb{1} + i \sum_{z,j}^{(\lambda, \mu)} \oplus \left(\frac{C^{SU(2)}(j)}{2a_1(F)} - \frac{C^{SU(2)}(j) + \frac{3}{4}y^2}{2a_{\frac{1}{2}}(F)} \right) \mathbb{1}_{z,j} . \quad (4.38) \end{aligned}$$

Here $\mathbb{1}$ is the unit matrix in the (λ, μ) irrep of $SU(3)$ and $\mathbb{1}_{z,j}$ are unit matrices in the $SU(2)$ irreps. Note that the inverse of the rotation represented by $D^{(\lambda, \mu)}(q)$ is denoted $D^{(\lambda, \mu)}(-q)$.

The field expression (4.25) is substituted in the Lagrangian density (3.9) in order to obtain the explicit expression in terms of collective coordinates and space coordinates. Some expressions with $SU(3)$ group generators, that are useful for this purpose, are presented in Appendix. After some lengthy manipulation the complete expression of the

Skyrme model Lagrangian density is obtained as:

$$\begin{aligned}
\mathcal{L}_{Sk} = & \frac{1}{4} \dim(\lambda, \mu) C_2^{SU(3)}(\lambda, \mu) \times \\
& \times \left\{ -\frac{(1 - \cos F)}{16} \left[f_\pi^2 + \frac{1}{4e^2} \left(F'^2 + \frac{2}{r^2} \sin^2 F \right) \right] \right. \\
& \quad \times \sum_{Z,M} (-1)^{Z+M} \left\{ \dot{q}^\alpha, C_\alpha'^{(Z, \frac{1}{2}, M)}(q) \right\} \left\{ \dot{q}^\beta, C_\beta'^{(Z, \frac{1}{2}, M)}(q) \right\} \\
& \quad - \frac{\sin^2 F}{8} \left[f_\pi^2 + \frac{1}{e^2} \left(F'^2 + \frac{1}{r^2} \sin^2 F \right) \right] \\
& \quad \times \sum_{Z,M} \left[(-1)^M \left\{ \dot{q}^\alpha, C_\alpha'^{(0,1,M)}(q) \right\} \left\{ \dot{q}^{\alpha'}, C_{\alpha'}'^{(0,1,M)}(q) \right\} \right. \\
& \quad \left. - \left(\left\{ \dot{q}^\alpha, C_\alpha''^{(0,1,\cdot)}(q) \right\} \cdot \hat{x} \right) \left(\left\{ \dot{q}^{\alpha'}, C_{\alpha'}''^{(0,1,\cdot)}(q) \right\} \cdot \hat{x} \right) \right] \\
& \left. - \mathcal{M}_{cl} - \Delta\mathcal{M}_1 - \Delta\mathcal{M}_2 - \Delta\mathcal{M}_3 - \Delta\mathcal{M}'(q) \right\}. \tag{4.39}
\end{aligned}$$

Here the following notation has been introduced:

$$\begin{aligned}
\Delta\mathcal{M}_1(F) = & -\frac{\sin^2 F}{30a_1^2(F)} \left[f_\pi^2 \left(12 \sin^2 F \cdot C_2^{SU(3)}(\lambda, \mu) - 16 \sin^2 F + 15 \right) \right. \\
& + \frac{1}{2e^2} \left(2F'^2 \left(12 \cos^2 F \cdot C_2^{SU(3)}(\lambda, \mu) + 16 \sin^2 F - 1 \right) \right. \\
& \left. \left. + \frac{\sin^2 F}{r^2} \left(6C_2^{SU(3)}(\lambda, \mu) + 7 \right) \right) \right], \tag{4.40a}
\end{aligned}$$

$$\begin{aligned}
\Delta\mathcal{M}_2(F) = & -\frac{(1 - \cos F)}{20a_{\frac{1}{2}}^2(F)} \left[f_\pi^2 \left(6(1 - \cos F) \cdot C_2^{SU(3)}(\lambda, \mu) + 3 \cos F + 2 \right) \right. \\
& \left. + \frac{1}{4e^2} \left(F'^2 \left(6(1 + \cos F) \cdot C_2^{SU(3)}(\lambda, \mu) - 3 \cos F + 2 \right) + 10 \frac{\sin^2 F}{r^2} \right) \right], \tag{4.40b}
\end{aligned}$$

$$\begin{aligned}
\Delta\mathcal{M}_3(F) = & -\frac{\sin^2 F}{30a_1(F) a_{\frac{1}{2}}(F)} \left[f_\pi^2 \left(12(1 - \cos F) \cdot C_2^{SU(3)}(\lambda, \mu) + 16 \cos F - 1 \right) \right. \\
& \left. + \frac{1}{2e^2} \left(F'^2 \left(4 \cos F \cdot \left(3C_2^{SU(3)}(\lambda, \mu) - 4 \right) + 15 \right) + 15 \frac{\sin^2 F}{r^2} \right) \right], \tag{4.40c}
\end{aligned}$$

$$\begin{aligned}
\Delta\mathcal{M}'(F, q) = & -\frac{3(1 - \cos F)}{16a_{\frac{1}{2}}^2(F)} \left[f_\pi^2 + \frac{1}{4e^2} \left(F'^2 + \frac{2}{r^2} \sin^2 F \right) \right] \\
& \times \left((-1)^{\bar{A}} C_{(\bar{A})}'^\alpha(q) C_\alpha'^{(0)}(q) C_{(-\bar{A})}'^\beta(q) C_\beta'^{(0)}(q) + 4 \right). \tag{4.40d}
\end{aligned}$$

[The “4” in the last bracket on the last row is missing in the corresponding expression in ref.[6], the consequence of which is the appearance of a spurious term $-3/8a_{1/2}(F)$ in eq. (69b) of that paper (there are some minor misprints in that equation as well)]. The notation (\bar{A}) indicates that only the states for which $I = \frac{1}{2}$ and $Z = \pm\frac{1}{2}$ are included. The $\Delta\mathcal{M}_k(F)$ terms may be interpreted as quantum mass corrections to the Lagrangian density. The $\Delta\mathcal{M}'(F, q)$ term depends on the quantum variables q^i and is an operator on the configuration space.

The integration (4.41) over the space variables and normalization by factor (3.15) gives the Lagrangian

$$\begin{aligned}
L_{Sk} &= \int \mathcal{L}_{Sk} d^3x = \frac{1}{8} \left\{ \dot{q}^i, C'_i{}^{(\bar{A})}(q) \right\} E_{(\bar{A})(\bar{B})} \left\{ \dot{q}^{i'}, C'^{(\bar{B})}_{i'}(q) \right\} \\
&\quad - M_{cl} - \Delta M_1 - \Delta M_2 - \Delta M_3 - \Delta M'(q) \\
&= -\frac{1}{8a_{\frac{1}{2}}(F)} (-1)^{\bar{A}} \left\{ \dot{q}^i, C'_i{}^{(\bar{A})}(q) \right\} \left\{ \dot{q}^{i'}, C'^{(-\bar{A})}_{i'}(q) \right\} \\
&\quad - \frac{1}{8} \left(\frac{1}{a_1(F)} - \frac{1}{a_{\frac{1}{2}}(F)} \right) (-1)^M \left\{ \dot{q}^i, C'^{(0,1,M)}_i(q) \right\} \left\{ \dot{q}^{i'}, C'^{(0,1,-M)}_{i'}(q) \right\} \\
&\quad - M_{cl} - \Delta M_1 - \Delta M_2 - \Delta M_3 - \Delta M'(q). \tag{4.41}
\end{aligned}$$

Here $M_{cl} = \frac{f_\pi}{e} \tilde{M}_{cl} = \int d^3x \mathcal{M}_{cl}(F)$, $\Delta M_k = e^3 f_\pi \Delta \tilde{M}_k = \int d^3x \Delta \mathcal{M}_k(F)$ and $\Delta M'(q) = \int d^3x \Delta \mathcal{M}'(q)$, where \tilde{M}_{cl} and \tilde{M}_k are integrals over the dimensionless variable.

5. STRUCTURE OF THE LAGRANGIAN AND THE HAMILTONIAN

The Wess-Zumino-Witten (WZW) action is given as an integral over the five dimensional manifold M^5 , the boundary of which is the compactified spacetime: $\partial M^5 = M^4 = S^3 \times S^1$. This term is necessary to account for the anomalies in QCD [14]. The standard form for this term is:

$$S_{WZ}(U) = -\frac{iN_c}{240\pi^2 N'} \int_{M^5} d^5x \epsilon^{\mu\nu\lambda\rho\sigma} \text{Tr} R_\mu R_\nu R_\lambda R_\rho R_\sigma, \tag{5.42}$$

where N_c is the number of colors and N' is a normalization factor. The derivation of the contribution of the Wess-Zumino-Witten term to the effective Lagrangian in the framework of collective coordinate formalism is given in ref. [15]. By application of Stoke's

theorem it takes the following form in a general dimension:

$$\begin{aligned}
L_{WZ}(q, \dot{q}) &= -\frac{iN_c}{24\pi^2 N'} \int_{M^3} d^3x \epsilon^{mjk} \text{Tr} \left[(\partial_m U_0) U_0^\dagger (\partial_j U_0) U_0^\dagger (\partial_k U_0) U_0^\dagger J_{(0,0,0)}^{(1,1)} \right] \\
&\times \frac{1}{2} \{ \dot{q}^\alpha, C_\alpha'^{(0)}(q) \} \\
&= -\frac{iN_c}{2\sqrt{3}\pi^2 N'} \int_{M^3} d^3x \frac{\sin^2 F(r)}{r^2} F'(r) \sum_{z,j}^{(\lambda,\mu)} yj(j+1)(2j+1) \frac{1}{2} \{ \dot{q}^\alpha, C_\alpha'^{(0)}(q) \} \\
&= -\lambda' \frac{i}{2} \{ \dot{q}^\alpha, C_\alpha'^{(0)}(q) \} .
\end{aligned} \tag{5.43}$$

Here

$$\lambda' = \frac{\sqrt{3}N_c B}{40N'} \dim(\lambda, \mu) C_3^{SU(3)}(\lambda, \mu) . \tag{5.44}$$

The coefficient λ' depends on the representation (λ, μ) . For all self adjoint irreps $\lambda = \mu$ the WZW term vanishes. Following Witten [14] the normalization factor is chosen to be $N' = \dim(\lambda, \mu) C_3^{SU(3)}(\lambda, \mu)/20$ so that $\lambda' = N_c B/2\sqrt{3}$. In the fundamental representation $N' = 1$. Here the coefficient λ' only serves to constrain the states of the system. Because the cubic Casimir operator $C_3^{SU(3)}$ (3.23) vanishes in the self-adjoint representations, it follows that the WZW term (5.43) also vanishes in those representations.

The Lagrangian of the system, with inclusion of the WZW term is:

$$L' = L_{Sk} + L_{WZ} . \tag{5.45}$$

There are seven collective coordinates. The momenta p_α that are canonically conjugate to q^α are defined as

$$p_\alpha = \frac{\partial L'}{\partial \dot{q}^\alpha} = \frac{1}{2} \{ \dot{q}^\beta, g_{\beta\alpha} \} - i\lambda' C_\alpha'^{(0)}(q) . \tag{5.46}$$

These satisfy the canonical commutation relations (4.35). The WZW term may be considered as an external potential in the system [16]. The seven right transformation generators may be defined as:

$$\hat{R}_{(\bar{A})} = \frac{i}{2} \left\{ p_\alpha + \lambda' i C_\alpha'^{(0)}(q), C_{(\bar{A})}'^\alpha(q) \right\} = \frac{i}{2} \left\{ \dot{q}^\beta, C_\beta'^{(\bar{B})}(q) \right\} E_{(\bar{B})(\bar{A})} . \tag{5.47}$$

The commutation rules for the generators (5.47) and their action on the $D^{(\lambda,\mu)}$ matrices are given in Appendix. It is convenient to define an eighth transformation generator formally as [6]:

$$\hat{R}_{(0)} = -\lambda' . \tag{5.48}$$

The $SU(2)$ subalgebra of the operators $\hat{R}_{(0,1,M)}$ satisfies the standard $SU(2)$ commutation conditions and may be interpreted as spin generators (Appendix). The eight left transformation generators may be defined as:

$$\hat{L}_{(B)} = \frac{1}{2} \left\{ \hat{R}_{(A)}, D_{(A)(B)}^{(1,1)}(-q) \right\}. \quad (5.49)$$

The transformation properties and commutation relations for the left and right transformation generators are given in Appendix.

The effective Lagrangian, which includes the WZW term takes the form:

$$\begin{aligned} L_{eff} &= \frac{1}{2a_{\frac{1}{2}}(F)} (-1)^{\bar{A}} \hat{R}_{(\bar{A})} \hat{R}_{(-\bar{A})} + \left(\frac{1}{2a_1(F)} - \frac{1}{2a_{\frac{1}{2}}(F)} \right) \left(\hat{R}_{(0,1,\cdot)} \cdot \hat{R}_{(0,1,\cdot)} \right) \\ &\quad - \lambda' \frac{i}{2} \left\{ \dot{q}^\alpha, C'_\alpha{}^{(0)}(q) \right\} - M_{cl} - \Delta M_1 - \Delta M_2 - \Delta M_3 - \Delta M'(q) \\ &= \frac{1}{2a_{\frac{1}{2}}(F)} \left((-1)^A \hat{L}_{(A)} \hat{L}_{(-A)} - \lambda'^2 \right) + \left(\frac{1}{2a_1(F)} - \frac{1}{2a_{\frac{1}{2}}(F)} \right) \left(\hat{R}_{(0,1,\cdot)} \cdot \hat{R}_{(0,1,\cdot)} \right) \\ &\quad - \lambda' \frac{i}{2} \left\{ \dot{q}^\alpha, C'_\alpha{}^{(0)}(q) \right\} - \Delta M_1 - \Delta M_2 - \Delta M_3 - M_{cl}. \end{aligned} \quad (5.50)$$

Note that the $\Delta M'(q)$ term which depends on quantum variables due to introducing of left translation generators (see (A.13)) in the Lagrangian expression (5.50) vanishes.

For the purpose of obtaining Euler-Lagrange equations that are consistent with the canonical equation of motion of the Hamiltonian, the general method of quantization on a curved space developed by Sugano *et al.* [17] is employed, in which the following auxiliary function is introduced:

$$\begin{aligned} Z(q) &= -\frac{1}{16} f^{ab} f^{cd} f^{ek} (\partial_a g_{cd}) (\partial_b g_{ek}) - \frac{1}{4} \partial_a (f^{ab} f^{cd} \partial_b g_{cd}) - \frac{1}{4} \partial_a \partial_b f^{ab} \\ &= -\frac{1}{4} \partial_b C'_{(\bar{A})}{}^a(q) E^{(\bar{A})(\bar{B})} \partial_a C'_{(\bar{B})}{}^b(q) \\ &\quad + \frac{3}{16a_{\frac{1}{2}}(F)} \left((-1)^{\bar{A}} C'_a{}^{(0)}(q) C'^a{}_{(\bar{A})}{}^b(q) C'^b{}_{(-\bar{A})}{}^c(q) C'_c{}^{(0)}(q) + 4 \right). \end{aligned} \quad (5.51)$$

With this the covariant kinetic term may be defined as:

$$\begin{aligned} 2K &= \frac{1}{2} \left\{ p_\alpha + i\lambda C'_\alpha{}^{(0)}(q), \dot{q}^\alpha \right\} - Z(q) \\ &= \frac{1}{a_{\frac{1}{2}}(F)} \left((-1)^A \hat{L}_{(A)} \hat{L}_{(-A)} - \lambda'^2 \right) + \left(\frac{1}{a_1(F)} - \frac{1}{a_{\frac{1}{2}}(F)} \right) \left(\hat{R}_{(0,1,\cdot)} \cdot \hat{R}_{(0,1,\cdot)} \right). \end{aligned} \quad (5.52)$$

According to the prescription [17] the effective Hamiltonian (with the constraint (5.48)) is constructed in the standard form as:

$$H = \frac{1}{2}\{p_\alpha, \dot{q}^\alpha\} - L_{eff} - Z(q) = K + \Delta M_1 + \Delta M_2 + \Delta M_3 + M_{cl}. \quad (5.53)$$

Upon renormalization the Lagrangian density (4.39) may be reexpressed in terms of left and right transformation generators. The effective Hamiltonian density without the symmetry breaking term in turn takes the form:

$$\begin{aligned} \mathcal{H}_{Sk} = & \frac{(1 - \cos F)}{4a_1^2(F)} \left[f_\pi^2 + \frac{1}{4e^2} \left(F'^2 + \frac{2}{r^2} \sin^2 F \right) \right] \\ & \times \left[(-1)^A \hat{L}_{(A)} \hat{L}_{(-A)} - \left(\hat{R}_{(0,1,\cdot)} \cdot \hat{R}_{(0,1,\cdot)} \right) - \lambda'^2 \right] \\ & + \frac{\sin^2 F}{2a_1^2(F)} \left[f_\pi^2 + \frac{1}{e^2} \left(F'^2 + \frac{1}{r^2} \sin^2 F \right) \right] \\ & \times \left[\left(\hat{R}_{(0,1,\cdot)} \cdot \hat{R}_{(0,1,\cdot)} \right) - \left(\hat{R}_{(0,1,\cdot)} \cdot \hat{x} \right) \left(\hat{R}_{(0,1,\cdot)} \cdot \hat{x} \right) \right] \\ & + \Delta \mathcal{M}_1 + \Delta \mathcal{M}_2 + \Delta \mathcal{M}_3 + \mathcal{M}_{cl}. \end{aligned} \quad (5.54)$$

The products of spin operators $\hat{R}_{(0,1,M)}$ may be separated into scalar and tensorial terms as:

$$\begin{aligned} & \left(\hat{R}_{(0,1,\cdot)} \cdot \hat{R}_{(0,1,\cdot)} \right) - \left(\hat{R}_{(0,1,\cdot)} \cdot \hat{x} \right) \left(\hat{R}_{(0,1,\cdot)} \cdot \hat{x} \right) = \\ & = \frac{2}{3} \left(\hat{R}_{(0,1,\cdot)} \cdot \hat{R}_{(0,1,\cdot)} \right) - \frac{4\pi}{3} Y_{2,M+M'}^*(\vartheta, \varphi) \left[\begin{matrix} 1 & 1 & 2 \\ M & M' & M+M' \end{matrix} \right] \hat{R}_{(0,1,M)} \hat{R}_{(0,1,M')}, \end{aligned} \quad (5.55)$$

where $Y_{l,M}(\vartheta, \varphi)$ is a spherical harmonic and the factor in the square brackets on the right-hand side is an SU(2) Clebsch-Gordan coefficient.

The covariant kinetic term (5.52) is a differential operator constructed from SU(3) left and SU(2) right transformation generators. The eigenstates of the Hamiltonian (5.53) are:

$$\left| \begin{matrix} (\Lambda, M) \\ Y, T, M_T; Y', S, M_S \end{matrix} \right\rangle = \sqrt{\dim(\Lambda, M)} D_{(Y, T, M_T)(Y', S, M_S)}^{*(\Lambda, M)}(q) |0\rangle. \quad (5.56)$$

Here the quantity D on the right-hand side is the complex conjugate Wigner matrix elements of (Λ, M) irrep of SU(3) in terms of quantum variables q^k . The topology of the eigenstates can be nontrivial and the quantum states contain an eighth "unphysical" quantum variable q^0 .

The matrix elements of the Hamiltonian density (5.54) for states with spin $S > \frac{1}{2}$ are not spherical and those states consequently have quadrupole moments. In the case

$S = \frac{1}{2}$ the matrix element of the second rank operator in right hand side of (5.55) vanishes.

6. THE SYMMETRY BREAKING MASS TERM

The chiral symmetry breaking mass term for the SU(3) soliton was defined (3.20). With the ansatz (4.25) in (3.20) the symmetry breaking density operator for the general irrep (λ, μ) obtains as:

$$\begin{aligned} \mathcal{L}_{SB} = -\mathcal{M}_{SB} = & -\frac{1}{N} \frac{f_\pi^2}{4} \left[m_0^2 \text{Tr} \left\{ U_0 + U_0^\dagger - 2\mathbb{1} \right\} \right. \\ & \left. - 2m_8^2 \text{Tr} \left\{ \left(U_0 + U_0^\dagger \right) J_{(0,0,0)}^{(1,1)} \right\} D_{(0)(0)}^{(1,1)}(-q) \right]. \end{aligned} \quad (6.57)$$

The operator (6.57) contains the matrix elements $D^{(1,1)}$, which depend on the quantum variables q^α . In this form this operator mixes the representations (Λ, M) of the eigenstates of the Hamiltonian [18]. The physical states of the system with symmetry breaking term therefore in principle have to be calculated by diagonalisation of the Hamiltonian. Since the mass term is minor part of the Lagrangian it may be considered as a perturbation in the SU(3) representation (Λ, M) .

For a given irrep (λ, μ) , in which the Lagrangian is defined, the symmetry breaking term depends on the chiral angle $F(r)$ as:

$$\begin{aligned} \text{Tr} \left\{ U_0 + U_0^\dagger - 2\mathbb{1} \right\} &= 2 \sum_{z,j}^{(\lambda,\mu)} \left(\sum_{m=-j}^j \cos 2mF(r) \right) - 2 \dim(\lambda, \mu) \\ &= 2 \frac{\sin(1+\lambda)F(r) + \sin(1+\mu)F(r) - \sin(\lambda+\mu+2)F(r)}{2 \sin F(r) - \sin 2F(r)} \\ &\quad - 2 \dim(\lambda, \mu). \end{aligned} \quad (6.58)$$

Further development of the expression (6.57) leads to:

$$\begin{aligned}
\text{Tr} \left\{ \left(U_0 + U_0^\dagger \right) J_{(0,0,0)}^{(1,1)} \right\} &= 2 \sum_{z,j}^{(\lambda,\mu)} 2\sqrt{3} \left[\frac{1}{3}(\lambda - \mu) + z \right] \left(\sum_{m=-j}^j \cos 2mF(r) \right) \\
&= \frac{2\sqrt{3}}{2 \sin F(r) - \sin 2F(r)} \\
&\quad \times \left\{ \frac{1}{2}(1 + \mu) (\sin(1 + \mu)F(r) - \sin(\lambda + \mu + 2)F(r)) \right. \\
&\quad + \frac{1}{3}(\lambda - \mu) (\sin(1 + \lambda)F(r) + \sin(1 + \mu)F(r) - \sin(\lambda + \mu + 2)F(r)) \\
&\quad + \frac{1}{2}(1 + \lambda) \left[(\sin F(r) - \sin(2 + \mu)F(r)) \cos \lambda F(r) \right. \\
&\quad \left. \left. - (\cos F(r) - \cos(2 + \mu)F(r)) \sin \lambda F(r) \right] \right\}. \tag{6.59}
\end{aligned}$$

For high irrep (λ, μ) the dependence of the symmetry breaking term on the chiral angle $F(r)$ differs significantly from that in the fundamental representation $(1, 0)$. In that representation the symmetry breaking term takes the standard form:

$$\mathcal{M}_{SB} = f_\pi^2 (1 - \cos F) \left[m_0^2 + \frac{1}{\sqrt{3}} m_8^2 D_{(0)(0)}^{(1,1)}(-q) \right]. \tag{6.60}$$

In the case of $(2, 0)$ representation the expression is:

$$\mathcal{M}_{SB} = \frac{1}{5} f_\pi^2 \left[(1 - \cos F + 2 \sin^2 F) m_0^2 - (1 - \cos F - 4 \sin^2 F) \frac{m_8^2}{\sqrt{3}} D_{(0)(0)}^{(1,1)}(-q) \right]. \tag{6.61}$$

Note that in both cases the asymptotical behavior at large distance of the symmetry breaking terms are different.

7. DISCUSSION

Above the SU(3) Skyrme model was quantized canonically in the framework of the collective coordinate formalism in for representations of arbitrary dimension. This lead to the complete quantum mechanical structure of the model on the homogeneous space SU(3)/U(1). The results extend those obtained earlier in the fundamental representation for SU(2) and SU(3) [5, 6] and those obtained in general representations of SU(2) [7, 8, 9]. The explicit representation dependence of the quantum corrections to the Skyrme model Lagrangian was derived. This dependence is nontrivial, especially for the Wess-Zumino-Witten and the symmetry breaking terms. The operators that form the Hamiltonian were shown to have well defined group-theoretical properties.

The choice of the irrep that is used for the unitary field depends on the phenomenological aspects of the physical system to which the model is applied. Formally the variation of the irrep can be interpreted as modification of the Skyrme model. The representation dependence of the Wess-Zumino-Witten term was shown to be absorbable into a normalization factor, with exception of the self adjoint irreps in which this term vanishes. The symmetry breaking term has different functional dependence on chiral angle $F(r)$ in different irreps. In case of self adjoint representations the symmetry breaking term, which is proportional to m_8^2 coefficient also vanishes.

The effective Hamiltonian (5.53) commutes with the left transformation generators $\hat{L}_{(A)}$ and the right transformation (spin) generators $\hat{R}_{(0,1,M)}$:

$$\left[\hat{L}_{(A)}, H \right] = \left[\hat{R}_{(0,1,M)}, H \right] = 0, \quad (7.62)$$

which ensures that the states (5.56) are the eigenstates of the effective Hamiltonian.

The symmetry breaking term does, however, not commute with the left generators:

$$\left[\hat{L}_{(Z, \frac{1}{2}, M)}, M_{SB} \right] \neq 0, \quad (7.63)$$

and therefore this term mixes the states in different representations (Λ, M) .

A new result of this investigation is the tensor term (5.56) in Hamiltonian density operator (5.55). Because of the tensor operator the states with spin $S > \frac{1}{2}$ have quadrupole moments.

Consider finally the energy functional of the quantum skyrmion in the states of (Λ, M) irrep. The problem is simplified if the symmetry breaking term that leads to representation mixing is dropped:

$$E(F) = \frac{C_2^{SU(3)}(\Lambda, M) - \lambda^2}{a_{\frac{1}{2}}(F)} + \left(\frac{1}{a_1(F)} - \frac{1}{a_{\frac{1}{2}}(F)} \right) S(S+1) + \Delta M_1 + \Delta M_2 + \Delta M_3 + M_{cl}. \quad (7.64)$$

The variational condition for the energy is:

$$\frac{\delta E(F)}{\delta F} = 0, \quad (7.65)$$

with the usual boundary conditions $F(0) = \pi, F(\infty) = 0$. At large distances this equation reduces to the asymptotic form

$$\tilde{r}^2 F'' + 2\tilde{r} F' - (2 + \tilde{m}^2 \tilde{r}^2) F = 0, \quad (7.66)$$

where the quantity \tilde{m}^2 is defined as:

$$\begin{aligned} \tilde{m}^2 = & -e^4 \left(\frac{1}{4\tilde{a}_1^2(F)} \left(C_2^{SU(3)}(\Lambda, M) - S(S+1) - \lambda'^2 + 1 \right) + \frac{2S(S+1) + 3}{3\tilde{a}_1^2(F)} \right. \\ & \left. + \frac{8\Delta\tilde{M}_1 + 4\Delta\tilde{M}_3}{3\tilde{a}_1(F)} + \frac{\Delta\tilde{M}_3 + 2\Delta\tilde{M}_2}{2\tilde{a}_1(F)} + \frac{1}{\tilde{a}_1(F)\tilde{a}_{\frac{1}{2}}(F)} \right). \end{aligned} \quad (7.67)$$

The corresponding asymptotic solution takes the form:

$$F(\tilde{r}) = k \left(\frac{\tilde{m}^2}{\tilde{r}} + \frac{1}{\tilde{r}^2} \right) \exp(-\tilde{m}\tilde{r}). \quad (7.68)$$

The quantum corrections depends on the irrep (λ, μ) to which the unitary field $U(\mathbf{x}, t)$ belongs as well as on the state irrep (Λ, M) and spin S . This bears on the stability of quantum skyrmion, the requirement of stability of which is that the integrals (4.33b, 4.33c) and ΔM_k converge. This requirement is satisfied only if $\tilde{m}^2 > 0$. That condition is only satisfied in the presence of the negative quantum mass corrections ΔM_k . It is the absence of this term, which leads to the instability of the skyrmion in the semiclassical approach [9] in the SU(2) case. Note that in the quantum treatment the chiral angle $F(\tilde{r})$ has the asymptotic exponential behavior (7.68) even in the chiral limit.

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APPENDIX A

The functions $C'_\alpha{}^{(\bar{A})}(q)$ defined in (2.2) are siebenbeins which constitute nonsingular 7×7 matrices. We can introduce the reciprocal functions $C'^\alpha_{(\bar{B})}(q)$ by:

$$\sum_{\bar{A}} C'_\alpha{}^{(\bar{A})}(q) \cdot C'^\beta_{(\bar{A})}(q) = \delta_{\alpha\beta}, \quad (\text{A.1a})$$

$$\sum_{\alpha} C'_\alpha{}^{(\bar{A})}(q) \cdot C'^\alpha_{(\bar{B})}(q) = \delta_{(\bar{A})(\bar{B})}. \quad (\text{A.1b})$$

Here (\bar{A}) and (\bar{B}) denote the basis of the irrep $(1, 1)$, with exception for the state $(0, 0, 0)$. The $C'_{(0)}^\alpha(q)$ are not defined.

The properties of the functions $C'_\alpha{}^{(K)}(q)$ follow from $\partial_\alpha \partial_\beta D^{(\lambda, \mu)} = \partial_\beta \partial_\alpha D^{(\lambda, \mu)}$:

$$\partial_\beta C'_\alpha{}^{(K)}(q) - \partial_\alpha C'_\beta{}^{(K)}(q) - \sqrt{3} \begin{bmatrix} (1,1) & (1,1) & (1,1)_a \\ (K') & (K'') & (K) \end{bmatrix} C'_\beta{}^{(K')}(q) C'_\alpha{}^{(K'')}(q) = 0, \quad (\text{A.2})$$

and are correct for all states (K) including $(0, 0, 0)$. The following properties of the functions $C'^\alpha_{(\bar{K})}(q)$ are useful:

$$\begin{aligned} & C'^\alpha_{(\bar{K}')} (q) \partial_\alpha C'^\beta_{(\bar{K}'')} (q) - C'^\alpha_{(\bar{K}'')} (q) \partial_\alpha C'^\beta_{(\bar{K}')} (q) + \sqrt{3} \begin{bmatrix} (1,1) & (1,1) & (1,1)_a \\ (\bar{K}') & (\bar{K}'') & (\bar{K}) \end{bmatrix} C'^\beta_{(\bar{K})} (q) \\ &= \sqrt{3} z'' C'^{(0)}_\alpha (q) C'^\alpha_{(\bar{K}')} (q) C'^\beta_{(\bar{K}'')} (q) - \sqrt{3} z' C'^{(0)}_\alpha (q) C'^\alpha_{(\bar{K}'')} (q) C'^\beta_{(\bar{K}')} (q). \end{aligned} \quad (\text{A.3})$$

In section 4 the right transformation generators (5.47) are defined with the following commutation relations:

$$\begin{aligned} \left[\hat{R}_{(\bar{A}')} , \hat{R}_{(\bar{A}'')} \right] &= -\sqrt{3} \begin{bmatrix} (1,1) & (1,1) & (1,1)_a \\ (\bar{A}') & (\bar{A}'') & (\bar{A}) \end{bmatrix} \hat{R}_{(\bar{A})} \\ &+ \sqrt{3} z'' \left\{ C'^\alpha_{(\bar{A}')} (q) C'^{(0)}_\alpha (q), \hat{R}_{(\bar{A}'')} \right\} - \sqrt{3} z' \left\{ C'^\alpha_{(\bar{A}'')} (q) C'^{(0)}_\alpha (q), \hat{R}_{(\bar{A}')} \right\}. \end{aligned} \quad (\text{A.4})$$

The $\text{SU}(2)$ subalgebra of the generators $\hat{R}_{(0,1,M)}$ satisfies the standard $\text{SU}(2)$ commutation relations. These may be interpreted as spin operators because its acting on unitary field (4.25) can be realized as a spatial rotation of skyrmion only:

$$\left[\hat{R}_{(0,1,M)}, A(q) U_0(x) A(q)^\dagger \right] = A(q) \left[J_{(0,1,M)}^{(1,1)}, U_0(x) \right] A^\dagger(q). \quad (\text{A.5})$$

The transformation rule for irrep matrices is:

$$\begin{aligned} \left[\hat{R}_{(\bar{K})}, D_{(A)(A')}^{(\lambda, \mu)}(q) \right] &= D_{(A)(A'')}^{(\lambda, \mu)}(q) \left\langle \begin{matrix} (\lambda, \mu) \\ A'' \end{matrix} \middle| J_{(\bar{K})}^{(1,1)} \middle| \begin{matrix} (\lambda, \mu) \\ A' \end{matrix} \right\rangle \\ &- \frac{\sqrt{3}}{2} y' C'^\alpha_{(\bar{K})}(q) C'^{(0)}_\alpha (q) D_{(A)(A')}^{(\lambda, \mu)}(q). \end{aligned} \quad (\text{A.6})$$

The eight left transformation generators are defined as:

$$\begin{aligned}\hat{L}_{(B)} &= \frac{1}{2} \left\{ \hat{R}_{(A)}, D_{(A)(B)}^{(1,1)}(-q) \right\} \\ &= \frac{i}{2} \left\{ p_\beta + \lambda i C_\beta'^{(0)}(q), K_{(B)}^\beta(q) \right\} + \lambda D_{(0)(B)}^{(1,1)}(-q),\end{aligned}\quad (\text{A.7})$$

where

$$K_{(B)}^\beta(q) = C_{(\bar{A})}^{\prime\beta}(q) D_{(\bar{A})(B)}^{(1,1)}(-q), \quad (\text{A.8})$$

the properties of which follows from (A.3):

$$K_{(B'')}^{\beta''}(q) \partial_{\beta''} K_{(B')}^{\beta'}(q) - K_{(B')}^{\beta''}(q) \partial_{\beta''} K_{(B'')}^{\beta'}(q) = \sqrt{3} \begin{bmatrix} (1,1) & (1,1) & (1,1)_a \\ (B'') & (B') & (B) \end{bmatrix} K_{(B)}^{\beta'}(q). \quad (\text{A.9})$$

By making use of (A.4) it may be proven that:

$$\left[\hat{L}_{(B')}, \hat{L}_{(B'')} \right] = \sqrt{3} \begin{bmatrix} (1,1) & (1,1) & (1,1)_a \\ (B') & (B'') & (B) \end{bmatrix} \hat{L}_{(B)}. \quad (\text{A.10})$$

The three right transformation generators or spin operators $\hat{R}_{(0,1,M)}$ commute with the left transformation generators:

$$\left[\hat{R}_{(0,1,M)}, \hat{L}_{(B)} \right] = 0. \quad (\text{A.11})$$

The left transformation rules for the irrep matrices are:

$$\begin{aligned}\left[\hat{L}_{(B)}, D_{(A')(A)}^{(\lambda,\mu)}(q) \right] &= \left\langle A' \left| J_{(B)}^{(1,1)} \right| A'' \right\rangle D_{(A'')(A)}^{(\lambda,\mu)}(q) - \frac{\sqrt{3}}{2} y_A \cdot D_{(0)(B)}^{(1,1)}(-q) D_{(A')(A)}^{(\lambda,\mu)}(q) \\ &\quad - \frac{\sqrt{3}}{2} y_A \cdot C_\alpha'^{(0)}(q) C_{(\bar{B}')}^{\prime\alpha}(q) D_{(\bar{B}')(B)}^{(1,1)}(-q) D_{(A')(A)}^{(\lambda,\mu)}(q).\end{aligned}\quad (\text{A.12})$$

It is straightforward to derive the following result:

$$\begin{aligned}(-1)^B \hat{L}_{(B)} \hat{L}_{(-B)} &= (-1)^{\bar{A}} \hat{R}_{(\bar{A})} \hat{R}_{(-\bar{A})} + \lambda'^2 - \frac{3}{4} \\ &\quad - \frac{3}{16} (-1)^{\bar{A}} C_\alpha'^{(0)}(q) C_{(\bar{A})}^{\prime\alpha}(q) C_\beta'^{(0)}(q) C_{(-\bar{A})}^{\prime\beta}(q).\end{aligned}\quad (\text{A.13})$$

For the derivation of the Lagrangian density the following expressions are needed:

$$\begin{aligned}E^{(\bar{A})(\bar{B})}(F) J_{(\bar{A})}^{(1,1)} J_{(\bar{B})}^{(1,1)} \\ = -\frac{1}{a_{\frac{1}{2}}(F)} \hat{C}_2^{SU(3)} + \left(\frac{1}{a_{\frac{1}{2}}(F)} - \frac{1}{a_1(F)} \right) \hat{C}^{SU(2)} + \frac{1}{a_{\frac{1}{2}}(F)} \left(J_{(0,0,0)}^{(1,1)} \right)^2,\end{aligned}\quad (\text{A.14})$$

$$\begin{aligned}
& E^{(\bar{A})(\bar{B})}(F) D_{(\bar{B}')(\bar{B})}^I(\hat{x}, F(r)) J_{(\bar{A})}^{(1,1)} J_{(\bar{B}')}^{(1,1)} \\
&= -\frac{\cos F}{a_{\frac{1}{2}}(F)} \hat{C}_2^{SU(3)} + \left(\frac{\cos F}{a_{\frac{1}{2}}(F)} - \frac{\cos 2F}{a_1(F)} \right) \hat{C}^{SU(2)} + \frac{\cos F}{a_{\frac{1}{2}}(F)} \left(J_{(0,0,0)}^{(1,1)} \right)^2 \\
&+ i \left(\frac{\sin 2F}{a_1(F)} + \frac{\sin F}{a_{\frac{1}{2}}(F)} \right) \left(J_{(0,1,\cdot)}^{(1,1)} \cdot \hat{x} \right) - 2 \frac{\sin^2 F}{a_1(F)} \left(J_{(0,1,\cdot)}^{(1,1)} \cdot \hat{x} \right) \left(J_{(0,1,\cdot)}^{(1,1)} \cdot \hat{x} \right). \quad (\text{A.15})
\end{aligned}$$

Here $D_{(\bar{B}')(\bar{B})}^I(\hat{x}, F(r))$ is a Wigner matrix of the $SU(2)$. The summation is over $SU(2)$ representations $I = \frac{1}{2}, 1$ and the corresponding bases,

$$E^{(\bar{A})(\bar{B})}(F) \begin{bmatrix} (1,1) & (1,1) & (1,1)_a \\ (\bar{A}) & (0,1,u) & (\bar{C}) \end{bmatrix} J_{(\bar{B})}^{(1,1)} J_{(\bar{C})}^{(1,1)} = \frac{1}{\sqrt{3}} \left(\frac{1}{2a_{\frac{1}{2}}(F)} + \frac{1}{a_1(F)} \right) J_{(0,1,u)}^{(1,1)}, \quad (\text{A.16})$$

$$\begin{aligned}
& E^{(\bar{A})(\bar{B})}(F) \begin{bmatrix} (1,1) & (1,1) & (1,1)_a \\ (\bar{A}) & (0,1,u) & (\bar{C}) \end{bmatrix} D_{(\bar{C}')(\bar{C})}^I(\hat{x}, F(r)) J_{(\bar{B})}^{(1,1)} J_{(\bar{C}')}^{(1,1)} \\
&= \frac{1}{\sqrt{3}} \left\{ - \left[J_{(0,1,\cdot)}^{(1,1)} \times \hat{x} \right]_u \left(\frac{\sin F}{2a_{\frac{1}{2}}(F)} + i \frac{2 \sin^2 F}{a_1(F)} \left(J_{(0,1,\cdot)}^{(1,1)} \cdot \hat{x} \right) \right) \right. \\
&\quad - i \left(\frac{\sin F}{2a_{\frac{1}{2}}(F)} \hat{C}_2^{SU(3)} - \left(\frac{\sin F}{2a_{\frac{1}{2}}(F)} - \frac{\sin 2F}{a_1(F)} \right) \hat{C}^{SU(2)} - \frac{\sin F}{2a_{\frac{1}{2}}(F)} \left(J_{(0,0,0)}^{(1,1)} \right)^2 \right) \hat{x}_u \\
&\quad \left. + \left(\frac{\cos 2F}{a_1(F)} + \frac{\cos F}{2a_{\frac{1}{2}}(F)} + i \frac{\sin 2F}{a_1(F)} \left(J_{(0,1,\cdot)}^{(1,1)} \cdot \hat{x} \right) \right) J_{(0,1,u)}^{(1,1)} \right\}. \quad (\text{A.17})
\end{aligned}$$

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- [1] T.H.R. Skyrme, Proc. Roy. Soc. **A 260**, 127 (1961).
 - [2] T.H.R. Skyrme, Nucl. Phys. **31**, 556 (1962)
 - [3] G.S. Adkins, C.R. Nappi and E. Witten, Nucl. Phys. **B 228**, 552(1983)
 - [4] H Walliser, Nucl. Phys. **A 548**, 649 (1992).
 - [5] K. Fujii, A. Kobushkin, K. Sato and N. Toyota, Phys. Rev. Lett. **58**, 651 (1987); Phys. Rev. **D 35**, 1896 (1987).
 - [6] K. Fujii, K. Sato and N. Toyota, Phys. Rev. **D 37**, 3663 (1988)
 - [7] E. Norvaišas and D.O. Riska, Physica Scripta **50**, 634 (1994).
 - [8] A. Acus, E. Norvaišas and D.O. Riska, Nucl. Phys. **A 614**, 361(1997).
 - [9] A. Acus, E. Norvaišas and D.O. Riska, Phys. Rev. **C 57**, 2597 (1998).
 - [10] S.L. Sondhi, A. Karlhede, S.A. Kivelson, E.H. Rezayi, Phys. Rev. **B 47**, 16419 (1993)

- [11] U.A. Khawaja and H. Stoof, Nature **411**, 918 (2001)
- [12] H. Yabu and K. Ando, Nucl. Phys. **B 301**, 601, (1988).
- [13] J.G. Kuriyan, D. Lurie and A.J. Macfarlane, J. Math. Phys. **6**, 722 (1965).
- [14] E. Witten, Nucl. Phys. **B 223**, 422 (1983); **B 223**, 433 (1983).
- [15] A.P. Balachandran, F. Lizzi, V.G.J. Rodgers, Nucl. Phys. **B 256**, 525 (1985).
- [16] P.O. Mazur, M.A. Nowak, and M. Praszalowicz, Phys. Lett. **B 147**, 137 (1984); L.C. Biedenharn, Y. Dothan, and A. Stern, Phys. Lett. **B 146**, 289 (1983); E. Guadagnini, Nucl. Phys. **B 236**, 35 (1984)
- [17] R. Sugano, Prog. Theor. Phys. **46**, 297 (1971); T. Kimura and R. Sugano, *ibid.* **47**, 1004 (1972); T. Kimura, T. Ohtani and R. Sugano, *ibid.* **48**, 1395 (1972); T. Ohtani and R. Sugano, *ibid.* **50**, 1715 (1973).
- [18] N. W. Park, J. Schechter and H. Weigel, Phys. Lett. **B224**, 171 (1989)